New kind of attractors in discontinuous piecewise linear maps

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We consider some families of discontinuous maps whose dynamics are characterized by the new kind of attractors called weird quasiperiodic attractors (WQA). These maps frequently appear in models describing the role of a regulator in economics and financial systems. Also applications in engineering have been observed.

Definition of the class. *We consider a discontinuous $n-$dimensional PWL map ($n \geq 1$) with a finite number of partitions separated by any kind of discontinuity set (of zero measure in the related space), with a unique fixed point.*

In the 1D case a one-dimensional analog of a WQA can be is a map with any finite number of discontinuity points, and sharing the same fixed point. We show that the only possible invariant sets of map G different from O are those occurring in a PWL circle map, i.e., intervals dense with nonhyperbolic periodic points or quasiperiodic orbits dense in some intervals, with possible coexistence.

A WQA A in 2D discontinuous map T appears as the closure of quasiperiodic trajectories, where the term "weird" refers to the rather complex and often intricate geometric structure of these attractors, and it is **a set satisfying the following properties**:

(i1) it is a topological attractor (i.e., it is a closed invariant set, $T(A) = A$, with a dense trajectory and an attracting neighborhood, any point of this neighborhood has A as $\omega-$limit);

(i2) any point of A has a quasiperiodic trajectory dense in A (i.e., there is no smaller invariant subsets of A, in particular, no periodic point);

(i3) the restriction of T to A cannot be reduced to a 1D map.

We prove the following

Theorem. *Let T be a 2D map as given in Definition. Then, the fixed point is the only cycle that can be hyperbolic and*

(j) a bounded ω-limit set A different from the fixed point, and from local invariant sets associated with it if nonhyperbolic, can only be one of the following kind::

(ja) it is a nonhyperbolic k-cycle, $k \geq 2$, belonging to k segments not intersecting any border, *filled with k-periodic points* (*all cycles have the same symbolic sequence)*;

(jb) it belongs to an invariant set on which the dynamics are reducible to a discontinuous 1D map;

(jc) it is a weird quasiperiodic attractor (WQA).

(jj) When A does not consist in segments filled with cycles, then map T exhibits (weak) sensitivity to initial conditions in A.

Thus a chaotic[1] set, on which the map is not reducible to a 1D map, cannot exist.

References

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[1] We define a chaotic set as a closed invariant set X having dense cycles and an aperiodic dense trajectory (meaning transitivity).